

AD649630

ON POSITIVE PRINCIPAL MINORS

BY

George B. Dantzig

TECHNICAL REPORT NO. 67-1

January 1967

ARCHIVE COPY

OPERATIONS
RESEARCH
HOUSE



Stanford
University
CALIFORNIA

DDC
RECEIVED
APR 7 1967
RECEIVED

ON POSITIVE PRINCIPAL MINORS

BY

George B. Dantzig

TECHNICAL REPORT NO. 67-1

January 1967

Operations Research House
Stanford University
Stanford, California

Research of G. B. Dantzig partially supported by Office of Naval Research, Contract ONR-N-00014-67-A-0112-0011, U. S. Atomic Energy Commission, Contract No. AT(04-3)-326 PA #18, and National Science Foundation Grant GP 6431; reproduction in whole or in part for any purpose of the United States Government is permitted.

ON POSITIVE PRINCIPAL MINORS

by

George B. Dantzig¹⁾

When a matrix is symmetric, the property of having all its principal minors positive, is equivalent to being positive definite. When a matrix is non-symmetric this is no longer true. For example

$$(1) \quad M = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$$

has all positive principal minors, but

$$xMx = x_1^2 - 7x_1x_2 + x_2^2 < 0 \text{ for } x = (x_1, x_2) = (1, 1)$$

The converse, however, as shown in Gale-Nikaido [1], Cottle [2] is true:

Theorem 1: The class with positive principal minors properly includes those which are positive definite.

Proof: Assume \bar{M} is positive definite, \bar{M} is non singular for if not, then there would exist $x = x^0 \neq 0$ such that $\bar{M}x^0 = 0$; yielding $x^{0T}\bar{M}x^0 = 0$, a contradiction. It follows that every principal submatrix of \bar{M} is also nonsingular.

¹⁾The author acknowledges leads suggested by David Gale.

Partition \bar{M} so that

$$\bar{M} = \begin{bmatrix} M_1 & c \\ r & a_{mm} \end{bmatrix}$$

choose $x^* = [Y^*, -1]$ where $Y^* M_1 = r$. Then

$$x^* \bar{M} x^* = [Y^*, -1] \begin{bmatrix} M_1 & c \\ r & a_{mm} \end{bmatrix} \begin{bmatrix} Y^* \\ -1 \end{bmatrix} = (a_{mm} - Y^* c) > 0$$

But $\det \begin{bmatrix} M_1 & c \\ r & a_{mm} \end{bmatrix} = \det \begin{bmatrix} M_1 & c \\ 0 & a_{mm} - Y^* c \end{bmatrix} = (\det M_1)(a_{mm} - Y^* c)$

Thus $\det \bar{M}$ and its largest order principal minor have the same sign. Inductively, since first-order principal minors are positive, so must the second order ones, etc., up to the highest order. Finally, example (1) shows that the inclusion is proper.

Although positive definite matrices \bar{M} do not comprise the entire class of positive principal minors, they can be used to generate a larger class by multiplying \bar{M} by diagonal matrices on the right and left to form $D\bar{M}E$. For example,

$$\begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/7 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 1 \end{bmatrix}$$

positive
definite

positive principal minors but
not positive definite

It is not difficult to show that for 2×2 matrices the entire positive principle minor class can be so obtained from the positive definite class. The following is easily seen.

Lemma: Let $D = [d_{ij}]$, $E = [e_{ij}]$ be diagonal matrices with the property that $d_{ii}e_{ii} > 0$, then for any M the sign (+, -, or 0) of any principal minor of DME is the same as that of M . If \bar{M} is positive definite, then \bar{DME} has positive principal minors.

Theorem 2: If M has positive principal minors and there exist diagonal D and E such that $\bar{M} = D^{-1}ME^{-1}$ is positive definite, then there exists a diagonal matrix F such that $F^{-1}MF$ is positive definite.

Proof: If \bar{M} is positive definite so is $\bar{A}M\bar{A}^T$ for any nonsingular A (since $x(\bar{A}M\bar{A}^T)x^T = yMy^T > 0$ where $x \neq 0$ and $y = xA \neq 0$). We consider $\bar{A}M\bar{A}^T = AD^{-1}ME^{-1}A^T$ and choose A so that $AD^{-1} = (E^{-1}A^T)^{-1}$ or $A^TA = ED$. Thus $A = [a_{ij}]$ could be chosen as a diagonal matrix such that $a_{ii} = +\sqrt{d_{ii}e_{ii}}$. Then $F = [f_{ij}]$ is the diagonal matrix $F = E^{-1}A^T$ where $f_{ii} = +\sqrt{d_{ii}/e_{ii}}$.

Theorem 3: The characteristic roots of (any) M are the same as $F^{-1}MF$ for (any) nonsingular F .²⁾

²⁾ A well known result of matrix theory.

Proof: By definition, λ is a characteristic root of M if there exists an $x \neq 0$ such that $Mx = \lambda x$. Setting $Fy = x$ then $y \neq 0$ if $x \neq 0$ and we have $MFy = \lambda Fy$ or $(F^{-1}MF)y = \lambda y$ so that λ is a characteristic root of $(F^{-1}MF)$ also.

Theorem 4: If \bar{M} is positive definite, the real part of every characteristic root is positive.³⁾

Proof: Let $\bar{M}x = \lambda x$, $x \neq 0$ and let $x = u + iv$, $\lambda = R + iS$, then by substitution and equating the real and imaginary parts

$$\bar{M}u = Ru - Sv \quad (u,v) \neq 0$$

$$\bar{M}v = Rv + Su$$

Now $R > 0$ follows from

$$0 < u^T \bar{M}u + v^T \bar{M}v = R(u^2 + v^2)$$

The following is known, see Gale - Nikaido [1].

Theorem 5: If M has positive principal minors, then every real characteristic root is positive.

³⁾ A well known result in matrix theory that is reviewed here for non-symmetric \bar{M} .

Proof: The characteristic equation is obtained by setting

$$\det [M - \lambda I] = 0$$

this yields

$$(-\lambda)^m + C_1(-\lambda)^{m-1} + C_2(-\lambda)^{m-2} + \dots + C_m = 0$$

where C_j is the sum of the j -th order principal minors. For matrices with positive principal minors, $C_j > 0$. It is not possible that $\lambda \leq 0$ because then all terms above would be non-negative and the last term positive so that the left hand side would be strictly positive, a contradiction.

Theorem 6 (Kalman*): There exist matrices M with positive principal minors that have characteristic roots not all of which have positive real parts; for such matrices no transformation $\bar{M} = DME$ exists with diagonal matrices D and E such that \bar{M} is positive definite.

* In a letter to D. Gale dated 9 July 1962, Rudolf E. Kalman showed that

$$\begin{bmatrix} 10/3 & -4 & -\sqrt{11.1} \\ 8/3 & 1/3 & -\sqrt{1.1} \\ -\sqrt{899.1} & -\sqrt{89.1} & 30 \end{bmatrix}$$

had positive minors but had complex characteristic roots with negative real parts.

Proof: If such D and E did exist, then by Theorem 2, an F would exist such that FMF^{-1} is positive definite. By Theorem 4 FMF^{-1} would have to have characteristic roots with positive real parts. By Theorem 3 the same would be true for M. However M in the example below does not have this property.

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 17 \\ 4 & 0 & 1 \end{bmatrix}$$

The first order principal minors are 1, 1, 1 whose sum $C_1 = 3$, its second order ones are 2, 1, 1 whose sum is $C_2 = 4$. Its third order one is $C_3 = 70$. Thus M has positive principal minors. Its characteristic equation is

$$\lambda^3 - C_1\lambda^2 + C_2\lambda - C_3 = 0$$

or
$$\lambda^3 - 3\lambda^2 + 4\lambda - 70 = 0$$

which factors into

$$(\lambda - 5)(\lambda^2 + 2\lambda + 14) = 0$$

The characteristic roots are

$$\lambda_1 = +5, \lambda_2 = -1 + i\sqrt{13}, \lambda_3 = -1 - i\sqrt{13}$$

Since the real parts of λ_2 and λ_3 are negative, this establishes that the class of positive principal minor matrices form a larger class

than those generated from the positive definite ones by simple rescaling of the rows and columns.

In mathematical programming one seeks p -vectors $x, y \geq 0$ that satisfy $y = Mx + q$ such that the products $x_i y_i = 0$ for $i = 1, 2, \dots, p$. The latter conditions may be replaced by the minimum value of $x^T y = x^T Mx + x^T q$, a quadratic function which is convex if and only if M is positive definite. Certain solution procedures based on convexity [2], [3], [4] turn out to be also valid even when M has positive principal minors. This leads to the speculation that by a simple change of units of x, y one could obtain a new system $\bar{y} = \bar{M}\bar{x} + \bar{q}$, $(\bar{y}, \bar{x}) \geq 0$, $\bar{y}_i \bar{x}_i = 0$ for $i = 1, \dots, p$ in which the quadratic function $\bar{x}^T \bar{M}\bar{x} + \bar{x}^T \bar{q}$ is convex. However we have shown that this is not always possible to do. The class of positive principal minor matrices does not appear to be a trivial extension of positive definite matrices. Solution techniques also valid for the latter have somehow gotten around the difficulties of local optimality usually associated with nonconvex programming problems.

An open question posed by Gale and Kalman and closely related to considerations found in a paper by Arrow and McManus [5] is the following:

Suppose for all diagonal matrices D such that $d_{ii} > 0$, that M has the property that the real part of every characteristic root of MD is positive, does this imply there exists a $D = D^0$ such that $D^0 M$ is positive definite?

REFERENCES

- [1] Gale, D. and Nakaido, H., "The Jacobian Matrix and Global Univalence of Mappings," Math. Ann. 159 (1965), 81-93.
- [2] Cottle, R., "Nonlinear Programs With Positively Bounded Jacobians," Ph.D. Thesis, University of California, Berkeley, March 1964.
- [3] Lemke, C. E., "Bimatrix Equilibrium Points and Mathematical Programming," Management Science 11 (1965), 681-689.
- [4] Dantzig, G. B. and Cottle, R. W., "Positive Semi-Definite Matrices and Mathematical Programming," University of California, Berkeley, Operations Research Center Report 63-18, 13 May 1963. Also in Non Linear Programming, (J. Abadie, ed.) North Holland, Amsterdam, 1966.
- [5] Arrow, K. J. and McManus, M., "A Note on Dynamic Stability," Econometrica 26 (1958) 448-454.
- [6] Kalman, R., Letter to D. Gale dated 9 July 1962 on stable matrices.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) Stanford University Operations Research House, Stanford, Calif.		2a. REPORT SECURITY CLASSIFICATION unclassified
		2b. GROUP
3. REPORT TITLE On Positive Principal Minors		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Technical Report		
5. AUTHOR(S) (Last name, first name, initial) Dantzig, George, B.		
6. REPORT DATE January 1967	7a. TOTAL NO. OF PAGES 8	7b. NO. OF REFS 6
8a. CONTRACT OR GRANT NO. N-00014-67-A-0112-0011 a. PROJECT NO. NR-047-064 c. d.	8a. ORIGINATOR'S REPORT NUMBER(S) Technical Report 67-1 8b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) Technical Report No. 67-1	
10. AVAILABILITY/LIMITATION NOTICES Distribution of this document is unlimited		
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch, Mathematical Sciences Division, Office of Naval Research, Washington, D. C. 20360
13. ABSTRACT The relation between matrices with all principal minors positive and positive definite matrices is explored in the non-symmetric case. It is shown that simple rescaling of rows and columns is insufficient to transform the former into the latter. As a consequence it appears that the class of problems that can be solved by complementary pivot theory of mathematical programming has been non-trivially extended beyond the convex case represented by linear and quadratic programming and takes its place along with another important extension of Lemke and Howson for the matrix associated with bi-matrix games.		

DD FORM 1473
1 JAN 64

unclassified
Security Classification

unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
<p>positive definite matrices, complementary pivot, mathematical programming, linear and quadratic programming bi-matrix games</p>						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (corporate author) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (either by the originator or by the sponsor), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those

imposed by security classification, using standard statements such as:

- (1) "Qualified requesters may obtain copies of this report from DDC."
- (2) "Foreign announcement and dissemination of this report by DDC is not authorized."
- (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through _____."
- (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through _____."
- (5) "All distribution of this report is controlled. Qualified DDC users shall request through _____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (paying for) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.